A universal framework of the homogenization problem of infinite dimensional diffusions (Stochastic Analysis of Jump Processes and Related Topics)

ALBEVERIO, Sergio; ROCKNER, Michael; YOSHIDA, Minoru W.

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A universal framework of the homogenization problem of infinite dimensional diffusions

Sergio ALBEVERIO [1] Michael RÖCKNER [2] and Minoru W. YOSHIDA [3]

Abstract

By generalizing the concrete formulations in [ABRY1,2], A universal frame work of the homogenization problem of infinite dimensional diffusions is proposed. The corresponding general structure is considered.

1 Probability space $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ on which the random coefficients are defined

Suppose that we are given the following:

$\{(\Theta_{\mathbf{k}}, \mathcal{B}_{\mathbf{k}}, \lambda_{\mathbf{k}})\}_{\mathbf{k}\in\mathbf{Z}^d}$: a system of complete probability spaces, where d is a given natural number.

$(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$: the probability space that is the completion of $(\prod_{\mathbf{k}} \Theta_{\mathbf{k}}, \bigotimes_{\mathbf{k}} \mathcal{B}_{\mathbf{k}}, \prod_{\mathbf{k}} \lambda_{\mathbf{k}})$, i.e., the completion of the direct product probability space.

$(\Theta, \overline{\mathcal{B}}, \mu)$: a complete probability space (corresponding to a Gibbs state) defined as follows: for $\forall D \subset\subset \mathbb{Z}^d$ and for any bounded measurable function φ defined on $\prod_{\mathbf{k}\in D'} \Theta_{\mathbf{k}}$ with some $\forall D' \subset\subset \mathbb{Z}^d$, μ satisfies

$$(\mathbb{E}^D \varphi, \mu) = (\varphi, \mu),$$

where

$$\begin{aligned}(\mathbb{E}^D \varphi)(\theta) &\equiv \int_{\Theta} \mathbb{E}^D(d\theta'|\theta) \\ &\equiv \int_{\Theta} \varphi(\theta'_D \cdot \theta_{D^c}) m_D(\theta'_D \cdot \theta_{D^c}) \overline{\lambda}(d\theta'),\end{aligned}$$

with

$$m_D(\theta'_D \cdot \theta_{D^c}) \equiv \frac{1}{Z_D(\theta_{D^c})} e^{-U_D(\theta'_D \cdot \theta_{D^c})}, \quad U_D \equiv \sum_{\mathbf{k}\in D} U_{\mathbf{k}},$$

$$\Theta \ni \theta \longmapsto \theta_D \in \prod_{\mathbf{k}\in D} \Theta_{\mathbf{k}}$$

[1] Inst. Angewandte Mathematik, Universität Bonn, Wegelerstr. 6,D-53115 Bonn (Germany), SFB611; BiBoS; CERFIM, Locarno; Acc. Architettura USI, Mendrisio; Ist. Mathematica, Università di Trento

[2] Dept. Math., Univ. Bielefeld

[3] e-mail wyoshida@ipcku.kansai-u.ac.jp fax +81 6 6330 3770. Kansai Univ., Dept. Mathematics, 564-8680 Yamate-Tyou 3-3-35 Suita Osaka(Japan)

is the natural projection, $\theta'_D \cdot \theta_{D^c}$ is the element $\theta'' \in \Theta$ such that

$$\theta''_D = \theta'_D, \qquad \theta''_{D^c} = \theta_{D^c},$$

also, for each $\mathbf{k} \in \mathbb{Z}^d$, $U_{\mathbf{k}}$ is a given bounded measurable function of which support is in $\prod_{|\mathbf{k}'-\mathbf{k}|\leq L} \Theta_{\mathbf{k}'}$, where the number L (the range of interactions) does not depend on $\mathbf{k}$, and $Z_D(\theta_{D^c})$ is the normalizing constant.

2 The ergodic flow

On $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ we are given an ergodic flow $T_{\mathbf{x}}$ (which is also a map on $(\Theta, \overline{\mathcal{B}}, \mu)$, but is not a measure preserving map on it) as follows:
Suppose that

$$\exists M_1 < \infty \quad \text{and} \quad \forall \mathbf{k} \in \mathbb{Z}^d \quad \text{there exists a } d_{\mathbf{k}} \text{ such that} \quad d_{\mathbf{k}} < M_1. \tag{2.1}$$

For each $\mathbf{x} \in \prod_{\mathbf{k}} \mathbb{R}^{d_{\mathbf{k}}}$ such that $\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with $\mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \ldots, x_{d_{\mathbf{k}}}^{\mathbf{k}})$
the map $T_{\mathbf{x}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\lambda})$ is defined by
i)

$$T_{\mathbf{x}} : \Theta \longrightarrow \Theta$$

that is a measure preserving transformation with respect to the measure $\overline{\lambda}$;
ii)

$$T_{\mathbf{0}} = \text{the identity},$$

$$\text{for } \mathbf{x},\, \mathbf{x}' \in \mathbf{x} \in \textstyle\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \qquad T_{\mathbf{x}+\mathbf{x}'} = T_{\mathbf{x}} \circ T_{\mathbf{x}'},$$

where

$$\mathbf{x} + \mathbf{x}' \equiv (\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d},$$

with

$$\mathbf{x}^{\mathbf{k}} + \mathbf{x}'^{\mathbf{k}} = (x_1^{\mathbf{k}} + x'^{\mathbf{k}}_1, \ldots, x_{d^{\mathbf{k}}}^{\mathbf{k}} + x'^{\mathbf{k}}_{d^{\mathbf{k}}}),$$

for

$$\mathbf{x} = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}^{\mathbf{k}} = (x_1^{\mathbf{k}}, \ldots, x_{d_{\mathbf{k}}}^{\mathbf{k}}),$$

$$\mathbf{x}' = (\mathbf{x}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{x}'^{\mathbf{k}} = (x'^{\mathbf{k}}_1, \ldots, x'^{\mathbf{k}}_{d_{\mathbf{k}}}),$$

and

$$\mathbf{0} \equiv (\mathbf{0}^{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, \qquad \mathbf{0}^{\mathbf{k}} = (0, \ldots, 0) \in \mathbb{R}^{d_{\mathbf{k}}};$$

iii)

$$(\mathbf{x}, \theta) \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \theta \longrightarrow T_{\mathbf{x}}(\theta) \in \Theta$$

is $\mathcal{B}(\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}) \times \overline{\mathcal{B}}/\overline{\mathcal{B}}$–measurable, where, $\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is assumed to be the topological space with the direct product topology;

iv) A function which is $T_{\mathbf{x}}$ invariant for all $\mathbf{x} \in \prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}$ is a constant function;

v) For $D \subset \mathbb{Z}^d$, let

$$\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}} \ni \mathbf{x} \longmapsto \mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$$

be the natural *projection*. If $\mathbf{x}_{D^c} = \mathbf{0}_{D^c}$, then

$$(T_{\mathbf{x}}(\theta))_{D^c} = \theta_{D^c}, \qquad \forall \theta \in \Theta, \quad \forall D \subset\subset \mathbb{Z}^d.$$

□

3 The core

We assume that an existence of a *core* $\mathcal{D}$. Namely, there exists $\mathcal{D}$ which is a dense subset of both $L^2(P)$ and $L^1(P)$, and $\forall \varphi \in \mathcal{D}$ satisfies
($\mathcal{D}$-1) φ is a bounded measurable function having only a finite number of variables θ_D for some $D \subset\subset \mathbb{Z}^d$,
($\mathcal{D}$-2)

$$\varphi(T_{\mathbf{x}_L}(\theta)) \in C^\infty(\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}} \to \mathbb{R}), \quad \forall \theta \in \Theta,$$

(cf. v) in the previous section) where we identify $\mathbf{x}_D \in \prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$ with an $\mathbf{x} \in (\prod_{\mathbf{k}\in\mathbb{Z}^d} \mathbb{R}^{d_{\mathbf{k}}})$ of which projection to $\prod_{\mathbf{k}\in D} \mathbb{R}^{d_{\mathbf{k}}}$ is $\mathbf{x}_D$,
($\mathcal{D}$-3) in ($\mathcal{D}$-2) for each $\theta \in \Theta$, all the partial derivatives of all orders of the function $\varphi(T_{\cdot}(\theta))$ (with the variables $\mathbf{x}_D$) are bounded and

$$\forall \varphi \in \mathcal{D}, \ \exists M < \infty; \quad |\nabla_{\mathbf{k}} \varphi(T_{\mathbf{x}}(\theta))| < M, \quad \forall \theta \in \Theta, \ \forall \mathbf{x}, \ \forall \mathbf{k} \in \mathbb{Z}^d, \tag{3.1}$$

where

$$\nabla_{\mathbf{k}} = (\frac{\partial}{x_1^{\mathbf{k}}}, \ldots, \frac{\partial}{x_{d_{\mathbf{k}}}^{\mathbf{k}}}).$$

□

4 Probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ on which the infinite dimensional diffusions are defined

Suppose that we are given a system of family of functions $a_{ij}^{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^d$, $1 \le i, j \le d_{\mathbf{k}}$ on $(\Theta, \overline{\mathcal{B}}, \overline{\mu})$ such that for each $\mathbf{k} \in \mathbb{Z}^d$ and each $1 \le i, j \le d_{\mathbf{k}}$, $a_{ij}^{\mathbf{k}}$ is a measurable function on $\Theta_{\mathbf{k}}$ and there exists $M_2 \in (0, \infty)$ and

$${M_2}^{-1} \le \sum_{1\le i,j\le d_{\mathbf{k}}} a_{ij}^{\mathbf{k}}(\theta_{\mathbf{k}}) x_i x_j \le M_2, \quad \forall \mathbf{k} \in \mathbb{Z}^d, \ \forall \theta_{\mathbf{k}} \in \Theta_{\mathbf{k}}, \ \forall (x_1, \ldots, x_{d_{\mathbf{k}}}) \in \mathbb{R}^{d_{\mathbf{k}}}. \tag{4.1}$$

We assume that

$$U_{\mathbf{k}}, \ a_{ij}^{\mathbf{k}} \in \mathcal{D}, \quad \mathbf{k} \in \mathbb{Z}^d, \ 1 \le i, j \le d_{\mathbf{k}}.$$

Also, we assume that there exists a common $M < \infty$ by which the evaluation (3.1) holds for all $a^{\mathbf{k}}_{i,j}$ and $U_{\mathbf{k}}$.

Finally, suppose that we are given a complete probability space $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $(t \in \mathbb{R}_+)$ with a filtoration $\mathcal{F}_t$. On $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ suppose that there exists a system of independent 1-dimensional $\mathcal{F}_t$-adapted Brownian motion processes

$$\{(B^{\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbf{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

Now, for each $\theta \in \Theta$, let

$$X^\theta \equiv \{(X^{\theta,\mathbf{k},i}(t))_{t\geq 0}\}_{\mathbf{k}\in\mathbf{Z}^d,\, 1\leq i\leq d_{\mathbf{k}}}.$$

be the unique solution of

$$\begin{aligned} X^{\theta,\mathbf{k},i}(t) = X^{\theta,\mathbf{k},i}(0) &+ \int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \{\frac{\partial}{\partial x^{\mathbf{k}}_j} a^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta)) \\ &-a^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))(\frac{\partial}{\partial x^{\mathbf{k}}_j}(\sum_{|\mathbf{k}-\mathbf{k}'|\leq L} U_{\mathbf{k}'}(T_{X^\theta(s)}(\theta))))\}ds \\ &+\int_0^t \sum_{1\leq j\leq d_{\mathbf{k}}} \sigma^{\mathbf{k}}_{ij}(T_{X^{\theta,\mathbf{k}}(s)}(\theta))dB^{\mathbf{k},j}(s), \qquad t\geq 0, \end{aligned} \tag{4.2}$$

where

$$(\sigma^{\mathbf{k}}_{ij}) = (2a^{\mathbf{k}}_{ij})^{\frac{1}{2}},$$

and

$$X^{\theta,\mathbf{k}}(t) = (X^{\theta,\mathbf{k},1}(t), \ldots, X^{\theta,\mathbf{k},d_{\mathbf{k}}}(t)),$$

also, by $X^\theta(t)$ we denote the vector

$$(X^{\theta,\mathbf{k}}(t))_{\mathbf{k}\in\mathbf{Z}^d} \in \prod_{\mathbf{k}\in\mathbf{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}.$$

Then, the random variable on $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ is the one taking values in

$$C([0,\infty) \to \prod_{\mathbf{k}\in\mathbf{Z}^d} \mathbb{R}^{d_{\mathbf{k}}}).$$

Through $(X^\theta(t))_{t\geq 0}$ we define a Θ valued process such that

$$\{T_{X^\theta(t)}(\theta)\}_{t\geq 0}.$$

Proposition 4.1 *The following hold:*
i) If $\quad T_{X^\theta(0)}(\theta) = T_{X^{\theta'}(0)}(\theta'), \quad P-a.s. \quad \omega\in\Omega,$
then

$$(X^\theta(t) - X^\theta(0))_{t\geq 0} = (X^{\theta'}(t) - X^{\theta'}(0))_{t\geq 0}, \qquad P-a.s. \quad \omega\in\Omega,$$

ii) For

$$\theta' = T_{X^\theta(0)}(\theta), \tag{4.3}$$

$$(T_{X^\theta(t)}(\theta))_{t\geq 0} = (T_{X_0^{\theta'}(t)}(\theta'))_{t\geq 0}, \qquad P-a.s. \quad \omega\in\Omega, \tag{4.4}$$

where $X_0^{\theta'}(t)(\theta')$ *is the diffusion defined by (4.2) with* $X_0^{\theta'}(0) = \mathbf{0}$ *and replacing* θ *by* θ' *in it.*
By (4.4) such $(T_{X^\theta(t)}(\theta))_{t\geq 0}$ *are represented by*

$$(Y_{\theta'}(t))_{t\geq 0} \equiv (T_{X_0^{\theta'}(t)}(\theta'))_{t\geq 0}$$

iii) *The process* $(Y_{\theta'}(t))_{t\geq 0})$ *satisfies* $Y_{\theta'}(0) = \theta'$ *and is a Markov process.*

□

Definition 4.1 *By Proposition 4.1, we define Markovian semi-groups corresponding to* $(X^\theta(t))_{t\geq 0}$ *and* $(Y_\cdot(t))_{t\geq 0}$*:*
For bounded measurable $f\in C(\prod_{\mathbf{k}\in D}\mathbb{R}^{d_\mathbf{k}}\to\mathbb{R})$ *with some dounded* $D\subset\subset\mathbb{Z}^d$*,*

$$(p_t^{X,\theta}f)(\mathbf{x}) \equiv E[f(X^\theta(t))\,|\,X^\theta(0)=\mathbf{x}], \qquad \mathbf{x}\in\prod_{\mathbf{k}}\mathbb{R}^{d_\mathbf{k}};$$

For $\varphi\in\mathcal{D}$

$$(p_t^Y\varphi)(\mathbf{y}) \equiv E[\varphi(T_{X^\theta(t)}(\theta))\,|\,T_{X^\theta(0)}(\theta)=\mathbf{y}], \qquad \mathbf{y}\in\Theta.$$

□

5 Key assumption and the result

As was done in [ABRY1,2], we assume the following:
There exist $K<\infty$ and $\gamma>0$ such that

$$\sup_{\mathbf{y}\in\Theta}|(p_t^Y\varphi)(\mathbf{y}) - <\varphi,\mu>| \leq K\,e^{-\gamma t}(\|\nabla\varphi\|_{L^\infty}+\|\varphi\|_{L^\infty}), \quad \forall\theta\in\Theta, \quad \forall\varphi\in\mathcal{D}, \tag{5.1}$$

where

$$\|\nabla\varphi\|_{L^\infty} = \sup_{\mathbf{x},\theta}|\nabla\varphi(T_\mathbf{x}(\theta))|, \qquad \|\varphi\|_{L^\infty} = \sup_\theta|\varphi(\theta)|.$$

□

Definition 5.1 *For each* $\mathbf{k}\in\mathbb{Z}^d$ *and* $i=1,\dots,d_\mathbf{k}$*, define an operator* $D^{\mathbf{k},i}:\mathcal{D}\to\mathcal{D}$ *such that*

$$(D^{\mathbf{k},i}\varphi)(\mathbf{y}) \equiv \frac{\partial}{\partial x_i^\mathbf{k}}\varphi(T_\mathbf{x}(\mathbf{y}))|_{\mathbf{x}=0}, \qquad \varphi\in\mathcal{D}, \quad \mathbf{y}\in\Theta.$$

□

Proposition 5.1 *For each* $\mathbf{k}\in\mathbb{Z}^d$*,* $i=1,\dots,d_\mathbf{k}$ *let*

$$b_i^\mathbf{k}(\mathbf{y}) \equiv \sum_{1\leq j\leq d_\mathbf{k}}\{(D^{\mathbf{k},j}a_{ij}^\mathbf{k})(\mathbf{y}) - a_{ij}^\mathbf{k}(\mathbf{y})(D^{\mathbf{k},j}(\sum_{|\mathbf{k}-\mathbf{k}'|\leq L}U_{\mathbf{k}'})(\mathbf{y}))\}, \quad \mathbf{y}\in\Theta,$$

then

$$\chi_i^{\theta,\mathbf{k}}(\mathbf{y}) \equiv \int_0^\infty (p_t^Y b_i^\mathbf{k})(\mathbf{y})dt \in L^2(\mu).$$

□

Through the same discussions performed in [ABRY1,2], for the present general framework we are also able to show the following (cf. [ABRY1,2] for the exact statement and the terminologies)

Theorem 5.1 *By taking a subsequence of the scaling process with the scaling parameter* $\epsilon > 0$ *such that* $\{\epsilon X^{\theta}(\frac{t}{\epsilon^2})\}_{t\geq 0}$, *for* $\mu - a.e.$ $\theta \in \Theta$ *it converges weakly to a Gaussian process with a constant covariance matrix, characterized by* $\sigma_{ij}^{\mathbf{k}}, \chi_i^{\theta,\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d, 1 \leq i, j \leq d_{\mathbf{k}}$, *and* $\theta \in \Theta$, *as* $\epsilon_n \downarrow 0$ *where* $\{\epsilon_n\}_{n\in\mathbb{N}}$ *is the sequence of the parameter corresponding to the subsequence of* $\{\epsilon X^{\theta}(\frac{t}{\epsilon^2})\}_{t\geq 0}$.

□

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